

# Unfolding Substructures of Complex Networks by Coupling Chaotic Oscillators beyond Global Synchronization Regime

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In the past decade, synchronization on complex networks has attracted increasing attentions from various research disciplines. Most previous works, however, focus only on the dynamic behaviors of synchronization process in the stable region, i.e., global synchronization. In this letter, we demonstrate that synchronization process on complex networks can efficiently reveal the substructures of networks when the coupling strength of chaotic oscillators is under the lower boundary of stable region. Both analytic and numerical results show that the nodes belonging to the same component in the hierarchical network are tightly clustered according to the Euclidean distances between the state vectors of the corresponding oscillators, and different levels of hierarchy can be systematically unfolded by gradually tuning the coupling strength. When the coupling strengths exceed the upper boundary of stable region, the hierarchy of the network cannot be recognized by this approach. Extensive simulations suggest that our method may provide a powerful tool to detect the hierarchical community structure of complex systems and deep insight into the relationship between structure and dynamics of complex systems.

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Synchronization phenomena of interacting units, such as clapping peoples, fireflies and pendulums, have been noticed by scientists for a long time [1–4]. For convenience of research, the interacting relations between units are usually abstracted as a network, whose nodes represent units and edges indicate interactions between them. With the dramatic progress made in complex network science in the last decade, researchers from various disciplines such as biology, physics and engineering communities have begun to explore a common interesting problem: the relation between topology of complex interactions and the emergent synchronization process [5–14]. On one hand, lots of works have investigated how synchronization process appears under different topology. For instance, the stability, one of the most important properties of synchronization, has been studied by the master stable function (MSF) method when the coupling networks have different topological properties like degree distribution, average path length, cluster coefficient and edge weight [15–21]. In most cases, the stability is proportional to the eigenratio  $\lambda_{max}/\lambda_2$ , where  $\lambda_{max}$  is the largest eigenvalue of Laplacian matrix of network and  $\lambda_2$  is the second smallest one. Based on the studies of stability, many methods are introduced to enhance the synchronizability via edge betweenness, topology modification, optimization, adaptive evolution, and so on [22–29]. On the other hand, few works focus on how topology of complex interactions can be reflected by synchronization

process. In [30], the hierarchical structure of network is gradually revealed from phase correlations in Kuramoto model [31] when the system evolves to the stable state. Recently, Ren *et.al* developed a universal approach to predict the exact network topology based solely on measuring the dynamical correlations of time series generated by the global synchronization [32].

In this letter, we focus on the problem of how substructures of complex networks can be reflected in the unstable synchronization processes taking place on networks. Both the analytical and numerical results show that if the coupling strength is under the lower boundary of stable region, the nodes belonging to different components of hierarchical structure can be directly distinguished by the state of the corresponding oscillator, which dramatically reduces the computational complexity of obtained cross-correlation among multiple time series in traditional methods. Moreover, for a hierarchical network, we can unravel different levels of hierarchies in a top-down (or bottom-up) way by decreasing (or increasing) coupling strength gradually. However, if the coupling strength exceeds the upper boundary of the stable region, we cannot find the cluster structures of nodes even they couple in a hierarchical topology. In the following, we will first show that the states of coupling oscillators and substructure of network topology are both associated with the Laplacian eigenvalues and eigenvectors in a theoretical way.

Consider a generic system composed of  $N$  coupling chaotic oscillators, each one is represented by a  $m$ -dimensional state vector  $\mathbf{x}_i$  and ruled by the differential equations  $\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i)$ . The network topology are defined by the Laplacian matrix  $\mathbf{L} = (l_{ij})_{N \times N}$ , in which  $l_{ij} = 1$  if oscillator  $i$  and  $j$  are coupled,  $l_{ij} = 0$  otherwise, and

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$l_{ii} = k_i$  (the degree of oscillator  $i$ ). Therefore, we have the evolution equation of system

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \sigma \sum_{j=1}^N l_{ij} \mathbf{H}(\mathbf{x}_j). \quad (1)$$

Here we only consider undirected and binary networks,

As the Laplacian matrix has zero row-sum, there is a global synchronization state of oscillators,

$$\mathbf{x}_1(t) \rightarrow \mathbf{x}_2(t) \rightarrow \dots \rightarrow \mathbf{x}_N(t) \rightarrow \mathbf{s}(t). \quad (2)$$

Let  $\delta \mathbf{x}_i = \mathbf{x}_i - \mathbf{s}$  be the deviation of the  $i$ th oscillator from the synchronization manifold, we have

$$\delta \dot{\mathbf{x}}_i = D\mathbf{F}(\mathbf{s})\delta \mathbf{x}_i - \sigma D\mathbf{H}(\mathbf{s}) \sum_{j=1}^N G_{ij} \delta \mathbf{x}_j. \quad (3)$$

Let  $\lambda_1 < \lambda_2 < \dots < \lambda_N$  denote Laplacian eigenvalues and  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N$  denote the associated Laplacian eigenvectors. To diagonalize the variational equations, we rewrite  $\delta \mathbf{x}$  as  $\delta \mathbf{x} = \mathbf{O}\xi$ , where  $\mathbf{O} = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N]$  and  $\xi = [\xi_1, \xi_2, \dots, \xi_N]^T$ . Hence, we obtain the  $N$  decoupled eigenmodes of variational equations

$$\dot{\xi}_i = [D\mathbf{F}(\mathbf{s}) - \sigma \lambda_i D\mathbf{H}(\mathbf{s})]\xi_i, i = 1, 2, \dots, N, \quad (4)$$

Eq. (4) is called the MSF of the complex system.  $\xi_i$  is the coefficient corresponding to eigenvector  $\mathbf{v}_i$  in the linear combination. The eigenvector  $\mathbf{v}_1 = [1, 1, \dots, 1]$  of  $\lambda_1 = 0$  corresponds to the synchronization manifold. The global synchronization of oscillators is achieved if  $\sigma \lambda_i$  ( $i \geq 2$ ) totally falls in the stable region  $\mathbf{S}$  which makes the maximum transverse Lyapunov exponents of Eq. (4) ( $i \geq 2$ ) be negative. The stable region  $\mathbf{S}$  can be empty( $\emptyset$ ), bounded( $[\alpha_1 \ \alpha_2]$ ) or unbounded( $[\alpha_1 \ \infty]$ ) with different chaotic oscillators and coupling function  $\mathbf{H}$ .

To identify different communities in a hierarchical network, we set coupling strength out of the stable region to ensure the differences between oscillator states remain. It's discovered that eigenvalues and eigenvectors of Laplacian matrix are strongly associated to substructures of network [33, 34]. In a network with  $m$  communities, the former  $m - 1$  non-trivial eigenvalues (from  $\lambda_2$  to  $\lambda_m$ ) are approximately equal to 0 and much smaller than the others, which leads to a gap between the former  $m$  eigenvalues and the others. And if a network is organized in a hierarchical way, there are more gaps between eigenvalues, which implies the different levels of hierarchy [30]. The eigenvectors of the former  $m - 1$  eigenvalues describe the organization of network. Elements of eigenvectors corresponding to nodes in the same community share approximately the same value. In the hierarchical network, the eigenvectors corresponding to the eigenvalues before the first gap only show a coarse macroscopic organization of the highest hierarchy. When eigenvalues increasing and crossing following gaps, their eigenvectors can suggest a finer mesoscopic organization of lower hierarchies. We

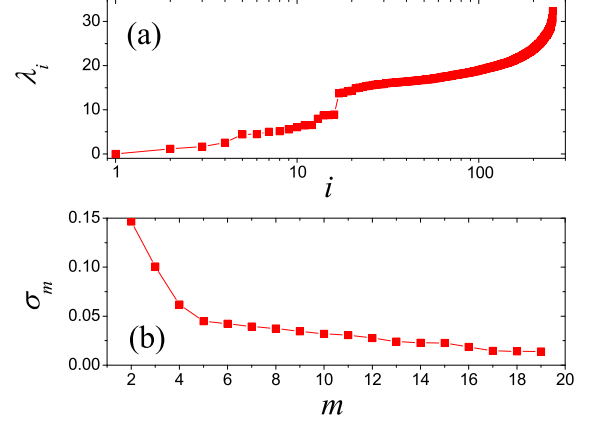


FIG. 1: (a)eigenvalues  $\lambda_i$  of Laplacian matrix of the hierarchical network considered.(b) Coupling strength  $\sigma_m$  versus  $m$ .

choose the coupling strength according to the inequality  $\sigma \lambda_m < \alpha_1 < \sigma \lambda_{m+1}$ , which makes the coefficients  $\xi_2$  to  $\xi_m$  diverge so that the final states of oscillators are linear combinations of eigenvectors  $\mathbf{v}_2$  to  $\mathbf{v}_m$  and the perturbations of  $\mathbf{v}_{m+1}$  to  $\mathbf{v}_N$  vanish for sufficient evolving time. By increasing  $m$ , the more eigenvectors corresponding to larger eigenvalues are added as the basis of the linear combination of state vectors and more details of hierarchical structure are uncovered by the state vectors.

We simulate the synchronization process of coupling chaotic oscillators on a hierarchical network to confirm the above theoretical analysis. We firstly choose the Rössler oscillators as an example of our investigation, whose stable region is bounded in  $[\alpha_1 \ \alpha_2]$ . Their state vectors and differential equations can be written as  $\mathbf{x} = [x, y, z]^T$  and  $\mathbf{F} = [-(y + z), x + 0.2y, 0.2 + z(x - 9)]^T$ . Units are coupled by the linear coupling function  $\mathbf{H}(\mathbf{x}) = [x, 0, 0]^T$ . We therefore can get  $\alpha_1 \approx 0.2$ , and  $\alpha_2 \approx 5$  according to Eq. 4. The hierarchical network is firstly proposed in [30], which includes two hierarchical levels of components. Specifically, it consists 256 nodes, in which 16 non-overlapping clusters each containing 16 nodes represent the first hierarchical level and 4 larger non-overlapping clusters each containing four ones from the first level consist the second hierarchical level. The connections of nodes at the first level ( $z_{in1}$ ), the second level ( $z_{in2}$ ) and with the rest nodes ( $z_{out}$ ) satisfy  $z_{in1} + z_{in2} + z_{out} = 18$ , and the hierarchical network is therefore indicated as  $z_{in1} - z_{in2}$ .

The synchronization process is performed on the 14-3 hierarchical network. The coupling strength is defined as

$$\sigma_m = \frac{1}{2} \left[ \frac{\alpha_1}{\lambda_m} + \frac{\alpha_1}{\lambda_{m+1}} \right], m = 2, 3, \dots, N - 1., \quad (5)$$

which guarantees  $\sigma \lambda_m < \alpha_1 < \sigma \lambda_{m+1}$  (i.e., the global synchronization is impossible). We show the Laplacian eigenvalues of the hierarchical network in Fig. 1(a), in

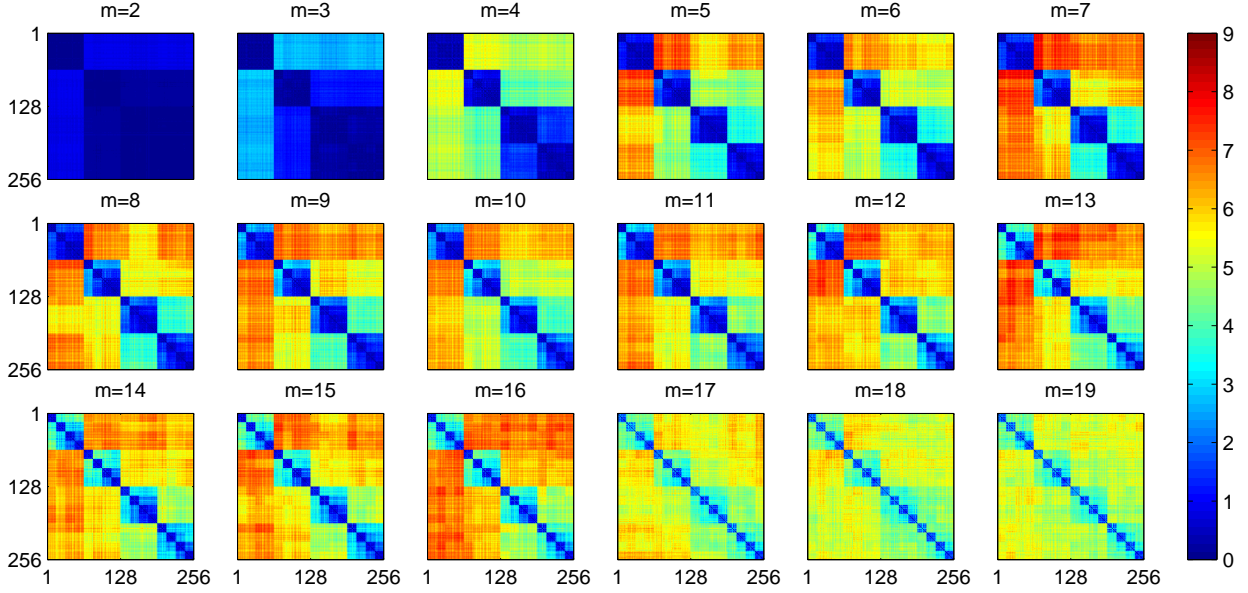


FIG. 2: (Color online) Distance matrix of 14-3 network with  $m$  changing from 2 to 19. With small  $m$ , only large clusters are recognized. Details of hierarchical structure emerge when the coupling strength decreases (i.e.,  $m$  increases).

which there are two obvious gaps that imply the two levels of hierarchy. As shown in Fig. 1(a), the first gap is between  $\lambda_4$  and  $\lambda_5$ , while the second one is between  $\lambda_{16}$  and  $\lambda_{17}$ . Thus, we are able to unfold the hierarchical structure gradually as  $m$  increases from 2 to 19. By considering the eigenvalues and lower limit of the stable region of Rössler oscillator, we set the coupling strength according to Eq. 5, which monotonously decreases with  $m$  (see in Fig. 1(b)). Moreover, the maximum coupling strength  $\sigma_2$  is also constrained by  $\sigma_2 \lambda_N < \alpha_2$  to make the coefficient  $\xi_i$  corresponding to  $\lambda_i (i \geq m+1)$  converge and perturbations of  $\mathbf{v}_{m+1}$  to  $\mathbf{v}_N$  vanish for sufficient evolving time.

We use the Euler method with time step  $\Delta t = 0.001$  and total evolving time  $T = 100$  to numerically simulate the synchronization process. The initial states of Rössler oscillators (i.e., the nodes of hierarchical network) are uniformly distributed in the interval  $[0, 1]$ . The difference of state between the nodes is suggested by their Euclidean distance  $d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}$ . Figure. 2 shows the distance matrices at  $T = 100$  at a changing coupling strength  $\sigma_m$  from  $m = 2$  to  $m = 19$ . Each distance matrix is averaged over a hundred runs of synchronization process. We find that the blocks consisting of trivial values exist in all distance matrices. Inside each block, the distances between nodes are small, while the distance between nodes across different blocks is large enough to unfold the hierarchy of network. Specifically, with smaller  $m$ , the coupling strength is large, which causes only the coefficients  $\xi_i$  corresponding to the smallest eigenvalues to diverge. In these cases, we can only recognize the larger clusters at the second hierarchical level.

When  $m \geq 5$ , the coefficients corresponding to the eigenvalues after the first gap begin to diverge, which suggests that in the distance matrices, the 4 non-overlapping components at the second hierarchical level split into smaller components at the first hierarchical level. As  $m$  increases further from 5 to 16, the eigenvectors corresponding to the larger eigenvalues can provide the difference between nodes that belong to the same component at the second hierarchical level, as is manifested in Fig 2. Thus, the 16 non-overlapping components at the second hierarchical level become much more obvious and the two-level hierarchy of network is gradually revealed. For  $m \geq 5$ , the coupling strength decrease to trivial values so that the synchronization process is weakened, which can help reveal more detailed structure of hierarchical network.

To show the clustering of nodes more explicitly, we present the dendrogram plots of agglomerative hierarchical trees and hierarchical structures implied by the hierarchical trees (see Fig. 3). The dendrogram plots of agglomerative hierarchical trees for  $m = 2, 3, 4$  and 16 are shown in Fig. 3 (a), (b), (c) and (d), respectively, and the corresponding hierarchical structures of network are described in Fig. 3 (e), (f), (g) and (h). When  $m = 2$ , the eigenvector  $v_2$  is dominant and the distances between nodes is to a large extent determined by it. The network in this case is partitioned into two asymmetric clusters, and one is much larger than the other one (corresponding to the distance matrix of  $m = 2$  in Fig. 2). Furthermore, when  $m$  increases from 2 to 4, the four non-overlapping components at the second level of hierarchical network emerge one by one. At last, the 16 non-overlapping components at the first level and 4 non-overlapping compo-

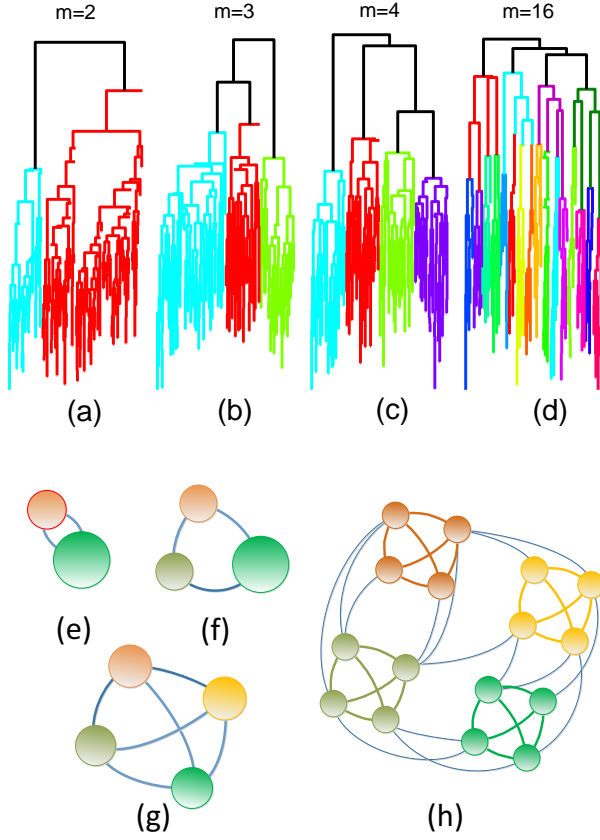


FIG. 3: (Color online) Plots ((a), (b), (c) and (d)) show the hierarchical tree, and plots ((e), (f), (g) and (h)) describe the cluster structure of networks implied by the hierarchical tree for  $m = 2, 3, 4$  and  $16$  respectively. It's demonstrated clearly that the hierarchical structure unfolded by synchronizing process change as coupling strength decreasing.

nents at second level are totally unfolded as the coefficients  $\xi_i$  corresponding to eigenvalues from  $v_2$  to  $v_{16}$  all diverge, as shown in Fig. 3(d) and (h).

We have demonstrated how the synchronization process of coupling chaotic oscillators can unravel the hierarchical structure of network at different levels by tuning the coupling strengths under the lower boundary of stable region. On the other hand, the global synchronization of Rössler oscillators is also broken if the coupling strengths exceed the upper boundary of stable region. In this situation, states vectors are linear combinations of eigenvectors corresponding to the largest eigenvalues, of which the elements are almost randomly distributed rather than clustered according to the substructures of network. Thus, the hierarchical structure of network cannot be revealed by the distance matrices. In the simulation, we set the coupling strength in agreement with the inequality  $\sigma\lambda_m < \alpha_2 < \sigma\lambda_{m+1}$  ( $m = 2, 3, \dots, N-1$ ). To eliminate the influence from the eigenvectors corresponding to small eigenvalues, the coupling strength is set to be larger than  $\alpha_1/\lambda_2 \simeq 0.17$ . For instance, if the cou-

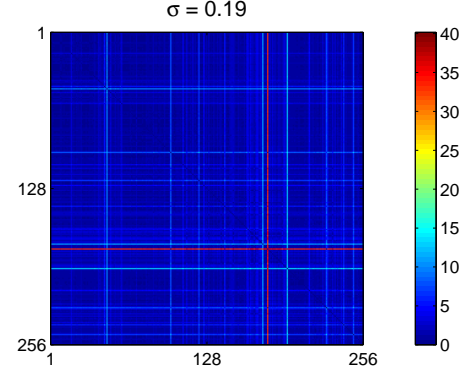


FIG. 4: (Color online) Distance matrix with coupling strength  $\sigma = 0.19$  at  $T = 100$ . No hierarchical structure can be seen in this matrix.

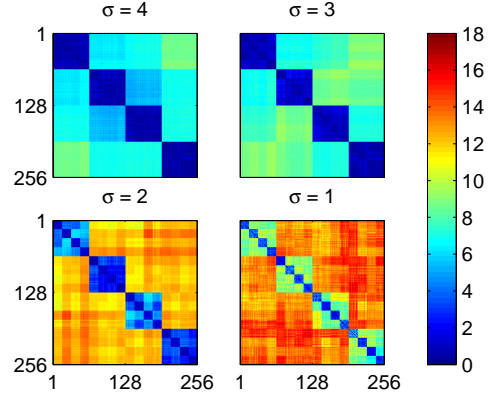


FIG. 5: (Color online) Distance matrix with coupling strength  $\sigma = 4, 3, 2$  and  $1$  at  $T = 100$ . As coupling strength decreasing, different levels of hierarchical structures are gradually revealed by synchronizing process

pling strength is chosen as  $\sigma = 0.19$ , the smallest value of  $m$  that satisfies  $\sigma\lambda_m < \alpha_2 < \sigma\lambda_{m+1}$  is  $m = 229$ , which suggests that  $\xi_i$  with  $i > 229$  diverges. The synchronization process performed on the same hierarchical network also evolves for time  $T = 100$ , and the distance matrix is also averaged over a hundred realizations. As shown in Fig. 4, the visualization of distance matrix shows no evidence of clustering of nodes. Most values in distances matrix are trivial, because almost all the states vectors are synchronized to the same state. Only few nodes are dramatically far away from the others. Therefore, although the global synchronization is also impossible because coupling strength exceeds the upper boundary of stable region, it does not provide useful information to unfold the hierarchical structure of networks.

We carried out the comprehensive numerical experiments using Lorenz oscillators to demonstrate the generality of our theoretical analysis. The state vector and differential equations of Lorenz oscillators can be written as

$\mathbf{x} = [x, y, z]^T$  and  $\mathbf{F} = [10(y - x), x(28 - z) - y, xy - \frac{8}{3}z]^T$  respectively. The units are also coupled by the linear coupling function  $\mathbf{H}(\mathbf{x}) = [x, 0, 0]^T$ . The stable region of linear coupling Lorenz oscillators is  $[\alpha_1 \infty]$ . Note that the numeric method gives  $\alpha_1 = 2$ , but the accurate value is much larger in the numeric simulation of synchronization process, and we therefore don't set the coupling strength according to Eq. 5. Furthermore, because the upper boundary of stable region of Lorenz oscillators is infinite ( $\infty$ ), we only need to consider the low boundary of the stable region, and vary the coupling strength  $\sigma$  from 4 to 1 for instance. The synchronization processes are performed on the 14-3 hierarchical network with the same conditions used in coupling Rössler oscillators. The distance matrices of nodes at  $T = 100$  are shown in Fig. 5. We can see that the coupling Lorenz oscillators on the hierarchical network beyond global synchronization region can gradually unfold the substructures and different hierarchical levels of the network.

In conclusion, we have both theoretically and numerically investigated the synchronization processes on hierarchical networks beyond stable region (i.e., the coupling strength is under the low boundary  $\alpha_1/\lambda_2$ ), which can be used to unfold the fine substructure such as different levels of hierarchies in an efficient way. The results

show that the nodes belonging to the same component are tightly clustered according to the state-vector distances between the corresponding oscillators. We also find that if the coupling strength exceeds the upper boundary  $\alpha_2/\lambda_N$ , the hierarchical structure of network cannot be effectively unfolded as the trajectories of the oscillators mix together. Since the nodes belonging to the same community can be directly identified by their state vectors, our approach can detect the hierarchical community structure of complex systems with very low computational complexity, compared to traditional cross-correlation based or spectral approaches. Our approach to unravel the hierarchical structure of network suggests that the dynamics of the interaction contains abundant information about the topology of the interaction, which can be utilized to infer the fine structure of complex system.

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